ALGEBRAIC NUMBER THEORY - FINAL TEST

4TH MAY, 2017, 02:00PM – 05:00PM

Instructions:

(i) *This exam is an open sheet exam – you can keep one* hand-written *A4 sized sheet with you as a cheat sheet (with anything written on it) for your reference.*
(ii) *The questions in section 1 are to be answered in True/False, no explanation need be given. All questions are compulsory. Each question carries 1 point.*
(iii) *In section 2 answer any three questions out of the four. In section 3 answer any two questions out of the three. Each question carries 9 points.*

1.

Question 1. *Answer in True/False. Each question carries 1 point.*

(1) *Let L_1, L_2 be Galois extensions of a field K and let $L = L_1L_2$. Then $L = L_1L_2$ is Galois over K and there is a natural isomorphism $Gal(L/K) \cong Gal(L_1/K) \times Gal(L_2/K)$.*
(2) *Let V/K be a finite dimensional vector space over a valued field $(K, |\cdot|)$, not necessarily complete, and $\|\cdot\|$ be a K-norm on V. Let $a_1, a_2, .., a_r$ be a basis of V. Let $v_n \in V$ be such that $\lim_{n\to\infty} v_n = 0$ for $\|\cdot\|$. If $v_n = \lambda_n^{(1)}a_1 + \cdots + \lambda_n^{(r)}a_r$ with $\lambda_n^{(i)} \in K$, then for all i, $\lim_{n\to\infty} \lambda_n^{(i)} \to 0$ for $|\cdot|$, for all i.*
(3) *Let L/K be a Galois extension of number fields. Then for any prime $\mathfrak{p}$ of K, there is a well defined Frobenius morphism $(\mathfrak{p}, L/K) \in Gal(L/K)$ provided $\mathfrak{p}$ is unramified in L.*
(4) *Let $\hat{G} := \{\chi \mid \chi : G \to \mathbb{C}^\times\}$ denote the the set of characters of a finite group G. Then $|\hat{G}| = |G|$ if and only if G is abelian.*
(5) *Let $\{S_n\}_{n=1}^{\infty}$ denote a collection of subsets of primes of a number field. Assume $S_n \subset S_{n+1}$ for all $n \geq 1$ and let $S = \cup_{n=1}^{\infty} S_n$. Let $\delta(\cdot)$ denote the Drichlet density function. Assume that $\delta(S_n)$ is defined for all $n \geq 1$ as well as $\delta(S)$ is well defined. Then we have that $\delta(S) = \sup\{\delta(S_n) \mid n \geq 1\}$, where* sup *denotes the supremum function.*

2.

Answer any three of the following. Each question carries 9 points.

Question 2. *Show that for any number field K, there is a finite extension L/K such that the extension of any integral ideal of K is principal in L.*

Question 3. *Let L/K be an extension of number fields. Show that for all but finitely many primes of L, the corresponding decomposition groups are cyclic.*

Question 4. *Let $K = \mathbb{Q}[\sqrt{-29}]$. For any prime $p \in \mathbb{Z}$ determine whether p remains inert, totally split or ramifies in K, in terms of a congruence condition on p.*

Question 5. *Let P_1, P_2 be sets of primes of a number field K. Assume that Drichlet density of P_1 resp. P_2 is well defined and is equal to $\delta(P_1)$ resp. $\delta(P_2)$. If $\delta(P_1) + \delta(P_2) > 1$, show that $|P_1 \cap P_2| = \infty$.*

3.

Answer any two of the following. Each question carries 9 points.

Question 6. *Compute class group of $\mathbb{Q}[\sqrt{-30}]$ as follows:*

(1) *Use Minkowski bound to show that primes $\mathfrak{p}$ that lie over (2), (3) and (5) generate the class group and calculate the decomposition of (2), (3) and (5) as a product of primes in $\mathbb{Q}[\sqrt{-30}]$.*
(2) *Use $N(\sqrt{-30}) = 30 = 2 \cdot 3 \cdot 5$ to compute the decomposition of the principal ideal $(\sqrt{-30})$ as $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are distinct prime ideals.*
(3) *Show that $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are not principal and use this to compute the class group.*

Question 7. *Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the non-archimedean valuation corresponding to prime p.*

(1) *Given any $e > 0$, show that there is an extension $\hat{L}/\mathbb{Q}_p$ which is totally ramified and of degree e.*
(2) *Show that any unramified extension $\hat{L}/\mathbb{Q}_p$ of degree d is of the form $\mathbb{Q}_p(\theta_m)$ where θ_m denotes a primitive m-th root of unity and $m = p^d - 1$.*

Question 8. *μ be the Möbius function defined by: $\mu(n) = 0$ if n is not square free, and $\mu(p_1...p_k) = (-1)^k$ where p_i are distinct primes. Show that the Drichlet series*

$$f(s) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

converges absolutely for $Re(s) > 1$ and in this region

$$f(s) = \frac{1}{\zeta(s)}$$

where $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ denotes the usual Riemann-zeta function.

Recall. A K-norm on a vector space V, where $(K, |\cdot|)$ is a valued field, is a norm $\|\cdot\|$ on V such that $\|\lambda v\| = |\lambda|\|v\|, \forall \lambda \in K, \forall v \in V$.

Minkowski Bound. For any fractional ideal $I \subset K$, there is an integral ideal $J \in [I]$ (where $[I]$ denotes the class of I in the class group) such that:

$$N_{K/\mathbb{Q}}(J) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|\Delta(K/\mathbb{Q})|}$$

where n is the degree of $K/\mathbb{Q}$, there are $2s$ complex embeddings of K in $\mathbb{C}$, and $\Delta(K/\mathbb{Q})$ is the discriminant of K.